Spectral Analysis of Boosted Space-Time Diversity Scheme

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Abstract—In this paper the asymptotic performance of a new intuitive space-time diversity scheme is analyzed. So called boosted scheme is compatible with today’s WLAN specifications with regard to convolutional coding and bit labelling, and minimizes the number of decoding iterations, required to obtain a reasonable Bit Error Rate. Good properties of the proposed scheme are proved by high asymptotic coding gain and advantageous distance spectrum. A simulation experiment is run to investigate the system performance in terms of poor channel state. The boosted scheme is compared with its ancestor – Bit-Interleaved Space-Time Coded Modulation with Iterative Decoding (BI-STCM-ID).

Index Terms—Multiple-input multiple-output channels, bit-interleaved space-time coded modulation, Alamouti scheme, constellation labeling, block fading, distance spectrum, coding gain.

I. INTRODUCTION

Wireless Local Area Networks have recently become a very popular Internet access technique. Almost each notebook is equipped with an 802.11 card. Expectations for WLAN throughput are still growing. The new 802.11n specification provides some promising techniques such as multi-antenna transmission with space-time block coding (STBC). The key issue is to make use of higher Multiple-Input Multiple-Output channel capacity. Reviewed in Section 2 BI-STCM-ID [2], that exploits iterative processing at the receiver, seems to be an excellent solution. Unfortunately, the 802.11n specification accepts only Gray constellation labelling, which seems to be an excellent solution. On the opposite, there are some constellation labellings, optimized for the lowest Bit Error Rate (BER) in case of error-free feedback. The author is an advocate of a new intuitive approach to an overall mapping (constellation labelling and space-time coding) described in Section 3. The proposed boosted space-time diversity scheme minimizes the number of passes while reasonable BER is kept. Theoretical analysis and simulation results are presented in Section 4. Section 5 of this paper is designed to conclude the work.

II. BI-STCM-ID OVERVIEW

A. System model

A BI-STCM-ID system is shown in Fig. 1. In the first instance, information bits are encoded by a convolutional encoder of rate \( R_C = 1/k_c \). Next, \( K \) interleaved encoded bits \([v_{1}^1 \ldots v_{K}^1] \) choose a vector \([x_{1}^1 \ldots x_{K}^1] \) of constellation points, each of them according to labelling rule \( \omega \). Next, \( q \) constellation points form the space-time (ST) symbol \( X_t \). The ST symbol consists of modified constellation points \( \{0,1\}^{K} \rightarrow \Re \) can be defined. In case of orthogonal \( 2 \times 2 \) Alamouti scheme, \( q = 2, L = 2 \), and

\[
X_t = \begin{bmatrix}
    x_1^1 \\
    -x_2^1 \\
    x_1^2 \\
    x_2^2
\end{bmatrix}
\]

(1)

The signals received by \( N_r \) antennas within \( L \) time slots are expressed by

\[
Y_t = X_t H_t + W_t
\]

(2)

Matrix \( H_t \) describes the channel, i.e. \([h_{k,j}]\) is a temporal gain of the path between \( i \)th transmit- and \( j \)th receive antenna. \( W_t \) represents the Gaussian noise.

Space-time demapper evaluates its output log-likelihood ratios (LLRs) \( \lambda (v_{k}^t) \) according to \( a \) priori LLRs \( \lambda (v_{k}^t) \) and the information received from the channel. SISO decoder [5] increases LLR’s reliability according to max-log-MAP routine.

B. BI-STCM-ID Asymptotic Performance

When ideal interleaving is assumed, the union bound of bit error probability is given by [3]:

\[
P_b \leq \frac{1}{K_c} \sum_{d=d_f}^{\infty} W_1(d) f (d, \omega, K),
\]

(3)

where \( d_f \) is the free distance of the convolutional code, and \( W_1(d) \) denotes the total input weight of error events at
Hamming distance $d$. Finally, $f(d, \omega, \mathcal{N})$ is the pairwise error probability (PEP). Its loosing Chernoff bound [3] is

$$f(d, \omega, \mathcal{N}) \leq \left[ \frac{1}{K2^R} \sum_{k=1}^{K} \sum_{b=0}^{1} \sum_{x \in \mathcal{N}^k_b} \sum_{z \in \mathcal{N}_b^k} \min \Phi_{\Delta(X, Z)}(s) \right]^{N_e}.$$  

where $Z$ is a “neighbor” of $X$, the label of which has opposite $k$th bit ($\bar{b}$ instead of $b$). $\Phi_{\Delta(X, Z)}(s)$ is the Laplace transform of probability density function

$$\Delta(X, Z) = \| Y - ZH \|^2 - \| Y - XH \|^2.$$  

Following [4], it can be written that

$$\min \Phi_{\Delta(X, Z)}(s) = \left[ \prod_{i=1}^{r} \left( 1 + \lambda_i / 4N_0 \right) \right]^{-N_e},$$  

where $\lambda_i$ are the nonzero eigenvalues of matrix

$$A = (X - Z)^H(X - Z),$$  

having rank $r$. Taking only the nearest neighbor $\tilde{Z} \in \mathcal{N}^k_b$ of $X$ in (4), one arrives at so-called expurgated PEP [3]:

$$f_{ex}(d, \omega, \mathcal{N}) \leq \left[ \frac{1}{K2^R} \sum_{k=1}^{K} \sum_{b=0}^{1} \sum_{x \in \mathcal{N}^k_b} \min \Phi_{\tilde{X}(Z)}(s) \right]^{N_e}.$$  

If $N_0 \rightarrow 0$,

$$f_{ex}(d, \omega, \mathcal{N}) \sim \left[ \frac{4}{\Omega^2 / N_0} \right]^{-\hat{r}N_e},$$  

where

$$\hat{\Omega}^2(\mathcal{N}, \omega, \mathcal{N}_e) = \left[ \frac{1}{K2^R} \sum_{k=1}^{K} \sum_{b=0}^{1} \sum_{x \in \mathcal{N}^k_b} \left( \prod_{i=1}^{r} \lambda_i \right)^{-N_e} \right]^{1/2}$$  

can be interpreted as an asymptotic coding gain associated with both space-time coding and constellation labeling. In the above statements $\hat{\lambda}$ and $\hat{r}$ are the nonzero eigenvalues and the rank of matrix $\hat{A} = (X - Z)^H(X - Z)$, respectively. (The expurgated PEP is accurate only for Gray-labelled schemes. In such case, there is exactly one nearest neighbor $Z$. For other labellings (7) is an overoptmistic approximation. [3])

Note that (8) is valid only for mapping rules $\omega$ with the same $\hat{r}$ value for each $(X, Z)$ pair. It has been checked that such condition is satisfied by the BI-STCM-ID with the Alamouti space-time code, considered in this paper.

Having taken only the first term (for $d = d_f$) in (3) and assumed that energy per information bit $E_b = 1/R$, where $R$ is the overall information rate, the BER for BI-STCM system (after the first pass or without iterative processing) is bounded on the logarithmic scale by [4]

$$\log_{10} \hat{P}_b \approx -\frac{\hat{r}N_e d_f}{10} \left[ (\hat{r}\hat{\Omega}^2)_{dB} + (E_b / N_0)_{dB} \right] + \text{const.}$$  

Note that the slope of the asymptotic bound is associated with the rank of $\hat{A}$. So only if all $\hat{A}$ matrixes (for each $(X, Z)$ pair) are full-ranked, full diversity gain can be reached. Additionally, the comparison of different mapping rules can be made only if the convolutional code of the same free distance $d_f$ is used. It is worth mentioning that the asymptotic coding gain $\hat{\Omega}^2$ of a mapping rule influences the horizontal offset of the bound (the higher coding gain, the better position of the asymptotic bound).

If the iterative decoding runs, one can assume the error-free feedback, i.e. all bits are assumed to be perfectly known at the demapper, except the one for which the LLR is currently being evaluated. In such case, BER is asymptotically bounded by

$$\log_{10} \hat{P}_b \approx -\frac{\hat{r}N_e d_f}{10} \left[ (\hat{r}\hat{\Omega}^2)_{dB} + (E_b / N_0)_{dB} \right] + \text{const.}$$  

where $\hat{\Omega}^2(\mathcal{N}, \omega, \mathcal{N}_e)$ is similar to $\hat{\Omega}^2(\mathcal{N}, \omega, \mathcal{N}_e)$ from (9), but $\hat{\lambda}$ and $\hat{r}$ must be replaced with $\lambda$ and $r$, that are respectively the nonzero eigenvalues and the rank of

$$\hat{A} = (X - Z)^H(X - Z).$$  

The bit labels of signals $X$ and $\tilde{Z}$ differ only on the $k$th bit position. Note that in the considered case there is exactly one $\tilde{Z}$ symbol for each $X$.

An accurate way to characterize labelling of Bit-Interleaved Coded Modulation with Iterative Decoding (an ancestor of BI-STCM-ID) is the Euclidean distance spectrum [6]. The idea is briefly depicted below. For each constellation point $x$ and each $k$-th position of its bit label, all neighboring points $z$ with the opposite $k$-th bit are found on the constellation. Distance spectrum $D$ is just a histogram of all $|x - z|^2$ entries. Such spectrum is proper to judge the asymptotic performance of the system without iterative processing. In the error-free feedback case, which can be approached after many iterations, $D_{ef}$ spectrum of $|x - z|^2$ distances should be evaluated, instead.

The interpretation of distance spectra is as follows. The lower frequency of short distances in $D$, the better asymptotic system performance after the first iteration. Similarly, low frequency of short distances in $D_{ef}$ suggests good asymptotic system performance in case of error-free feedback. Note that the spectrum analysis is useful to compare different mapping rules, and does not cover the impact of the employed convolutional code on overall system performance.

Let us extend the idea of distance spectrum for any space-time diversity scheme. If an orthogonal space-time code is used, the issue of the overall mapping rule $\omega$ optimization is reduced to search for optimal constellation labelling $\omega$. To find
this statement true, see Theorem 1 in [4]. As a more general approach, the author proposes to associate the spectrum $D$ with $\left(\prod_{i=1}^{r} \lambda_i\right)$ values. In the same manner $D_{o,f}$ should consist of $\left(\prod_{i=1}^{\tilde{r}} \lambda_i\right)$ values. The correspondence between the meaning of the Euclidean distance for 2-dimensional space and the meaning of the product of eigenvalues for matrices makes this approach justified. The idea of distance spectrum is utilized in Section 4 to examine the performance of the proposed space-time diversity scheme.

III. BOOSTED SPACE-TIME DIVERSITY SCHEME FOR WLAN SYSTEMS

The most common approach to BI-STCM-ID is to use an orthogonal STBC and find a constellation labelling $\omega$ to maximize coding gain $\Omega^2$. In the region of BI-STCM-ID potential applications, like WLAN systems, decoding time is the key issue. Unfortunately, any optimized labelling makes the convergence of iterative process slower [2]. The solution would be a new overall mapping rule $\omega$, thanks to which a demanded BER can be achieved after only a few iterations. The compatibility with the IEEE WLAN specifications would be highly appreciated.

The author proposed in [7] an intuitive space-time diversity scheme for WLAN systems, which is shown in Fig. 2. The convolutional encoder is taken from 802.11a/g/n specifications ([171 133]OCT). The idea is to take advantage of both Gray and “optimal” [4] labellings of 16-QAM. There are two signal streams at the transmitter. The first one, with the Gray mapper, is expected to provide good performance after the first decoding pass. The second one is responsible for high asymptotic coding gain. (The author has proved in [8] that $A$ matrices are full-ranked for each $(X, Z)$ pair, i.e. $\tilde{r} = 2$. Therefore, eq. (11) remains valid.)

The shaded blocks in Fig. 2 are used optionally, and should be turned off when other devices in a network run in legacy mode. The block denoted by $\Pi$ is a symbol interleaver of unitary depth. The resultant space-time codeword is

$$X_t = \begin{bmatrix} x_1^1 (\text{Gray}) & x_1^2 (\text{Gray}) \\
 x_2^1 (\text{opt.}) & x_2^2 (\text{opt.}) \end{bmatrix}.$$  

Fig. 3 represents the distance spectra $D$ of Gray- and “optimally”-labelled BI-STCM-ID and the boosted space-time diversity scheme. As it can be noticed, the spectrum of the boosted scheme is the worst one, i.e. the highest frequency of the lowest possible entry occurs. Moreover, the CDF of the proposed scheme grows much faster than for both BI-STCM-ID systems, considered in this paper. So it can be concluded that the spectrum of the boosted scheme contains many low-valued entries. The best mapping rule in this competition is the Gray-labelled BI-STCM-ID with its slowly increasing CDF.

Note that poor asymptotic performance of the boosted scheme does not result in slow convergence of iterative process. The last can be examined by means of EXtrinsic Information Transfer (EXIT) chart [9]. The author of this paper showed in [8] that convergence of the boosted scheme is very

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IV. EVALUATION OF BOOSTED SPACE-TIME DIVERSITY SCHEME

Let us analyze the distance spectra $D$ and $D_{o,f}$ of the boosted system and BI-STCM-ID – the latter with both “optimal” and Gray labellings. For convenience, spectrum entries can be scaled by the shortest possible distance $d_0$, as in [6] for BICM-ID. (In that paper the entries were written as multiplicities of the minimum squared Euclidean distance $|x - z|^2$ between a constellation point $x$ and its nearest neighbor $z$).

For better legibility, entries $d/d_0$ of the spectra will be treated as values of random variable $D$, whose cumulative distribution function $Pr(D < d/d_0)$ will be plotted instead of the original spectrum. Note that the abscissa will be scaled logarithmically.

Fig. 3}

![Fig. 3: Distance spectra $D$](image1.png)

![Fig. 4: Distance spectra $D_{o,f}$](image2.png)
TABLE I
ASYMPTOTIC CODING GAIN FOR DIFFERENT MAPPING RULES

<table>
<thead>
<tr>
<th>OVERALL MAPPING RULE</th>
<th>Asymptotic coding gain $\Omega^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gray-labelled BI-STCM-ID</td>
<td>0.4298</td>
</tr>
<tr>
<td>Optimally-labelled BI-STCM-ID</td>
<td>2.3414</td>
</tr>
<tr>
<td>Boosted space-time diversity scheme</td>
<td>1.0896</td>
</tr>
</tbody>
</table>

Fig. 5. Bit Error Rate vs. $E_b/N_0$

fast, i.e. the improvement in the system performance from one iteration to another is significant.

The boosted scheme involves iterative decoding. Therefore, its advantages should occur in the error-free feedback case. The distance spectra $D_{\text{eff}}$ of all considered systems are plotted in Fig. 4. As it was said above, to obtain good asymptotic performance, the frequency of short distances in the spectrum should be minimized. The shortest distance occurring in the spectrum should be maximized as well. In light of these assumptions, the Gray-labelled BI-STCM-ID is the worst one (most of the spectrum entries have the lowest possible value $d/d_0 = 1$). Therefore, such system is inappropriate for iterative decoding.

The boosted scheme is far better than the Gray-labelled BI-STCM-ID, as the shortest possible distance $d/d_0 = 1$ does not occur at all. Instead, the most common entry $d/d_0 = 5$ accounts for as much as $3/8$ of the total, and the highest $d/d_0$ (one in every four entries) equals $90$.

The “optimally”-labelled BI-STCM-ID wins the competition for the best mapping rule in the error-free feedback case. The lowest distance (every other entry) is $d/d_0 = 25$, and the highest (one in every four) equals $160$.

A “compact” quality measure of a mapping rule is the asymptotic coding gain $\Omega^2$. Its values for all the considered systems are listed in Table I. The results confirm the analysis of distance spectrum, i.e. the Gray-labelled BI-STCM-ID is the worst system under the condition of error-free feedback, the “optimally”-labelled BI-STCM-ID is the best one, and the boosted scheme is in the middle.

The mapping rule is not the only one that influences the whole system performance. To work properly, the system requires a good match between the mapping rule and the convolutional code. The author of this paper showed in [7] that the “optimally”-labelled BI-STCM-ID, in contrary to the boosted scheme, cannot cooperate with the $[171\ 133]_{\text{OCT}}$ code at low $E_b/N_0$ values (i.e. the decoding trajectory on the EXIT chart is “pinched off”). In fact, “optimally”-labelled BI-STCM-ID has only been considered in the literature in combination with convolutional codes of short free distance to avoid the pinch-off effect.

Thanks to the fact that the boosted scheme is well matched to the $[171\ 133]_{\text{OCT}}$ code, two goals are achieved: compatibility with the 802.11n specification, and the asymptotic bound steeper than for “optimally”-labelled BI-STCM-ID. In consequence, the latter performs worse asymptotically, in spite of higher asymptotic coding gain.

Till now it has been shown that the boosted space-time diversity scheme performs better asymptotically than the Gray-labelled BI-STCM-ID. To compare the performance at low $E_b/N_0$, Monte Carlo simulation was conducted. Each frame consisted of $10^8$ data bits were transmitted for each $E_b/N_0$ value. The simulation results are shown in Table I. It can be observed that the decrement in Bit Error Rate from one iteration to another is insignificant for Gray-labelled BI-STCM-ID, which makes the iterative processing worthless. The first-pass performance of the boosted scheme is worse than for BI-STCM-ID. This fact results from disadvantageous $D$ spectrum of the boosted scheme. Nevertheless, the iterative process converges fast and a reasonable BER can be reached after only a few iterations.

V. CONCLUSION

A boosted scheme deriving advantages from both Gray and “optimal” constellation labellings has been analyzed. The proposed scheme outperforms the Gray-labelled BI-STCM-ID for any $E_b/N_0$ value. The “optimally”-labelled BI-STCM-ID has been excepted from the comparison due to a mismatch between optimal labelling and $[171\ 133]_{\text{OCT}}$ convolutional code.

As orthogonality of the space-time code in the proposed scheme has been lost, signal detection is more complex and further research on its simplification is necessary.

REFERENCES


Maciej Krasicki received the M.S. degree in Electronics and Telecommunications from Poznan University of Technology, Poland, in 2006. Since then he has been working towards the Ph.D. degree. His dissertation work concerns a new (“boosted”) space-time diversity scheme, designed to support iterative decoding at the receiver of WLAN systems. His Ph.D. defense took place in 2010.

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